CONVERGENCE IN PROBABILITY OF RANDOM POWER SERIES AND A RELATED PROBLEM IN LINEAR TOPOLOGICAL SPACES(1)

BY

LUDWIG ARNOLD

ABSTRACT

A linear topological space is said to have the circle property if every power series with coefficients in it has a circle of convergence. Every complete locally convex or locally bounded space has the circle property, but not a certain class of F-spaces including the space of all random variables on a non-atomic probability space, endowed with the topology of convergence in probability.

1. **Introduction.** Let (Ω, F, P) be a probability space and $\{a_n(\omega)\}_{n=0}^{\infty}$ an arbitrary sequence of complex-valued random variables defined on it. The formal power series

$$F(z, \omega) = \sum_{n=0}^{\infty} a_n(\omega) z^n$$

where z is an element of the complex plane C, is called a *random power series*. Such a series is said to *converge* (in any mode considered in probability theory) at the point z if the sequence of its partial sums converges at z.

Recently [1, 2] we gave an example of a random power series converging in probability only at the points $z = 0$ and $z = 1$, and nowhere else(2).

Therefore, in general, for the convergence in probability of a random power series there exists no so-called *circle of convergence* (i.e. a circle around $z = 0$ such that we have convergence inside but divergence outside). On the other hand, such a circle always exists for almost sure convergence and convergence in the pth mean ($p > 0$).

The first aim of this note is to characterize the class of probability spaces (Ω, F, P), for which every random power series which can be defined on it has a circle of convergence in probability.

Furthermore, the set $M(\Omega, F, P)$ of all equivalence classes of complex-valued random variables defined on (Ω, F, P), endowed with the topology of convergence

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(2) Professor H. Rubin pointed out that the constructions given in [1], p. 86 and [2], p. 6 can be generalized to give a random power series converging in probability at $z = 0$ and in a prescribed denumerable set of complex numbers having no finite limit point, but nowhere else.

in probability, forms a linear topological space, in particular an F-space (see e.g. [4], p. 329). This leads our attention to the power series

$$f(z) = \sum_{n=0}^{\infty} a_n z^n \tag{1}$$

whose coefficients a_n are elements of an arbitrary linear topological space X over the complex field C, and $z \in C$. Such a space X is said to have the *circle property*, if every power series (1) possesses a circle of convergence. For instance, every Banach space has the circle property ([9], or [4] pp. 224–232), and in this case the circle of convergence of (1) has radius $(\limsup \sqrt[n]{\| a_n \|})^{-1}$.

Our second aim is to give some sufficient conditions for the circle property and to describe a class of spaces which fail to have this property.

2. **Probability spaces with the circle property.** Let (Ω, F, P) be an arbitrary probability space. A set $A \in F$ is called an *atom* if $P(A) > 0$, and if $B \in F$, $B \subset A$, then either $P(B) = P(A)$ or $P(B) = 0$. If $\{A_n\}$ is the (at most countable) family of disjoint atoms of (Ω, F, P) and if $P(\Omega - \cup A_n) = 0$, the probability space is called *atomic*.

THEOREM 1. *Every random power series with coefficients defined on a fixed probability space (Ω, F, P) has a circle of convergence in probability if and only if (Ω, F, P) is atomic.*

Proof. (a) Suppose (Ω, F, P) is atomic. Then convergence in probability is equivalent to almost sure convergence. But for the latter there always exists a circle of convergence.

(b) Suppose $P(B) > 0$ where $B = \Omega - \cup A_n$. In this case we can construct a random power series without a circle of convergence in probability. To avoid redundance, let us assume that $B = \Omega$.

By a theorem of S. Saks (see [4], p. 308), for every $\varepsilon > 0$ there exist finitely many disjoint sets $B_1, \cdots, B_m \in F$ with $\cup B_j = \Omega$ and $0 < P(B_j) \leqq \varepsilon$. We set $\varepsilon = \varepsilon_k$ where $\varepsilon_k > 0$, $\varepsilon_k \to 0 (k \to \infty)$, and arrange the elements of the resulting partitions of Ω for $k = 1, 2, \cdots$ in a sequence $\{C_n\}$.

Now let $s_0(\omega) = 0 \, \forall \omega$ and

$$s_n(\omega) = n^n I_{C_n}(\omega) \qquad (n \geqq 1),$$

where I_A denotes the indicator function of a set A. The random power series $\sum s_n z^n$ cannot converge in probability at any $z \neq 0$. For, if $z \neq 0$ is fixed and $C_{n_0}, \cdots, C_{n_1} (n_0 \leqq \cdots \leqq n_1)$ are the elements of a complete partition of Ω with $(n_0 | z |)^{n_0} \geqq 1$, we have

$$P\left\{\omega \,\Big|\, \left| \sum_{n_0}^{n_1} s_n(\omega) z^n \right| \geqq 1 \right\} = P\Omega = 1.$$

Now let us consider the series $\Sigma a_n z^n$ where $a_0(\omega) = 0 \, \forall \omega$ and $a_n = s_n - s_{n-1}$ ($n \geqq 1$). We have

$$\sum_0^n a_k z^k = s_n z^n + (1-z) \sum_0^{n-1} s_k z^k$$

and $s_n z^n \to 0$ in probability $\forall z \, (n \to \infty)$, since $\varepsilon_k \to 0$.

Hence, $\Sigma a_n z^n$ converges in probability at $z = 0$ and $z = 1$ but at no other point. Otherwise

$$(1-z)^{-1} \left(\sum_0^n a_k z^k - s_n z^n \right) = \sum_0^{n-1} s_k z^k$$

would converge, in contradiction to what was proved above, q.e.d.

3. **Power series with coefficients in a linear topological space.** As mentioned above, in the F-space $M = M(\Omega, F, P)$ with norm

$$\| x \| = E \frac{|x(\omega)|}{1 + |x(\omega)|} = \int_\Omega \frac{|x(\omega)|}{1 + |x(\omega)|} dP$$

we have $\| x_n - x \| \to 0$ if and only if $x_n \to x$ in probability.

So considered, Theorem 1 states that M has the circle property if and only if it is isomorphic either to an Euclidean space (in the case of finitely many atoms) or to the F-space (s) of all complex sequences $c = (c_1, c_2, \cdots)$ with the norm

$$\| c \| = \sum_{n=1}^\infty \frac{1}{2^n} \frac{|c_n|}{1 + |c_n|}$$

(in the case of countably many atoms).

Now let us consider the general case (1). *Ordinary convergence* (O) of a series Σx_n, $x_n \in X$, is defined as usual. We follow A. Dvoretzky [5] and call Σx_n *absolutely convergent* (A) if

$$\sum_n p_V(x_n) < \infty$$

for every neighborhood V of $0 \in X$, where $p_V(x)$ is the Minkowski functional of V at x, i.e.

$$p_V(x) = \inf \{ \lambda \mid \lambda > 0, \; x \in \lambda V \}.$$

In a linear metric space with norm $\| x \|$, the series Σx_n is said to *converge metrically* (M) if $\Sigma \| x_n \| < \infty$. In such a space we have $M \Rightarrow A$, in Banach spaces $M \Leftrightarrow A$, in F-spaces $M \Rightarrow O$.

In general, in an arbitrary linear topological space A does not entail O, and O does not entail A. But, for a power series (1) we have the following simple facts:

LEMMA 1. *Let X be a linear topological space and* (1) *a power series in X.*
(a) *There always exists a circle of convergence for A.*
(b) *O at z_1 (or only $\{a_n z_1^n\}$ bounded) $\Rightarrow A$ at every z with $|z| < |z_1|$.*

Proof. (a) For each linear topological space the family W of circled[3] neighborhoods of 0 is a local base (see [6], p. 35), so that it is sufficient to consider W. For every $U \in W$

$$p_U(\alpha x) = |\alpha| \, p_U(x) \, \forall \alpha \in C$$

see [6], p. 15). Therefore

$$\Sigma p_U(a_n z^n) = \Sigma p_U(a_n) |z|^n$$

converges in a circle with radius $(\limsup \sqrt[n]{p_U(a_n)})^{-1}$, and altogether we have A for

$$|z| < r(A) = \inf_{U \in W} (\limsup \sqrt[n]{p_U(a)})^{-1},$$

and absolute divergence of (1) $\forall |z| > r(A)$.

(b) O at z_1 implies $a_n z_1^n \to 0$. A convergent sequence in a linear topological space is bounded, so that for every $U \in W$ there exists an $\varepsilon > 0$ such that $\alpha a_n z_1^n \in U \, \forall n$ whenever $|\alpha| \leqq \varepsilon$. Thus, for $|z| < |z_1|$

$$\Sigma p_U(a_n z^n) = \Sigma p_U\left(a_n z_1^n \left(\frac{z}{z_1}\right)^n\right) = \Sigma p_U(a_n z_1^n) \left|\frac{z}{z_1}\right|^n \leqq \frac{1}{\varepsilon} \Sigma \left|\frac{z}{z_1}\right|^n < \infty,$$

q.e.d.

We denote by $r(A)$ the radius of the A-circle, and if X has the circle property (for O), we denote the corresponding radius by $r(O)$.

LEMMA 2. *Let $X = \Pi_{t \in T} X_t$ be the Cartesian product of linear topological spaces X_t each having the circle property with $r^{(t)}(O) = r^{(t)}(A) = r_t \, \forall t \in T$. Then X also has the circle property with*

$$r(O) = r(A) = \inf_{t \in T} r_t.$$

Proof. These statements follow at once from the fact that a series Σx_n in X, $x_n = (\cdots, x_n^{(t)}, \cdots)$, converges $O(A)$ if and only if $\Sigma x_n^{(t)}$ converges $O(A) \forall t \in T$, q.e.d.

Lemma 2 already covers the atomic case of Theorem 1. Lemma 1(b) states that one part of the classical situation can be found in every linear topological

(3) A set V in a linear space is circled if $aV \subset V$ whenever $|a| \leqq 1$.

space. The other part, $A \Rightarrow O$, would, together with Lemma 1(b), imply the circle property and $r(O) = r(A)$. But for this we need additional conditions. For instance, every finite-dimensional space has the circle property with $r(O) = r(A)$. Other classes will be described below.

4. **Complete locally convex spaces.**

THEOREM 2. *If X is a complete locally convex space, then $A \Rightarrow O$, so that X has the circle property and $r(O) = r(A)$ for every power series* (1).

Proof. $A \Rightarrow O$ follows easily from the fact that the Minkowski functional of a circled convex body is a pseudonorm and X is complete. The rest follows from Lemma 1, q.e.d.

It should be mentioned that it is possible to develop a satisfactory theory of holomorphic functions defined in a domain $G \subset C$ with values in a complete locally convex space (see also [7]).

5. **Complete locally bounded spaces.** Each locally bounded space (i.e. a space possesing an open bounded set) can be endowed with a p-homogeneous norm $\| x \|$ (i.e. $\| \alpha x \| = |\alpha|^p \| x \|$ $\alpha \in C$ where $0 < p \leqq 1$) reproducing the original topology (see [8]). Conversely, a linear metric space with a p-homogeneous norm is locally bounded.

THEOREM 3. *If X is a complete locally bounded space and if $r(M)$ denotes the radius of M-convergence of* (1) *with respect to a* (*always existing*) *p-homogeneous norm $\| \cdot \|$, X has the circle property with*

$$r(O) = r(A) = r(M) = (\limsup \sqrt[n]{\| a_n \|})^{-1/p}.$$

Proof. Clearly, by the p-homogenity of $\| \cdot \|$, we have $r(M) = (\limsup \sqrt[n]{\| a_n \|})^{-1/p}$. Taking into account the relations among O, A, and M stated in section 3 it remains to prove $r(A) = r(M)$. This follows from the fact that for a p-homogeneous norm $\Sigma \| x_n \|^{1/p} < \infty$ is a necessary and sufficient condition for A-convergence of Σx_n, q.e.d.

Consider an arbitrary F-space X with norm $\| \cdot \|$. Without loss of the generality, we can always assume that $\| \alpha x \| \leqq \| x \|$ for $|\alpha| \leqq 1$, so that the domain of M-convergence is a circle. The function

$$\phi(\alpha) = \sup_{\substack{x \in X \\ x \neq 0}} \frac{\| \alpha x \|}{\| x \|} \qquad (\alpha \geqq 0)$$

has the properties $\phi(0) = 0$, $\phi(1) = 1$, $\phi(\alpha + \beta) \leqq \phi(\alpha) + \phi(\beta)$, $\phi(\alpha\beta) \leqq \phi(\alpha)\phi(\beta)$, and is increasing with α. Clearly, since $\phi(\alpha^2) \leqq \phi(\alpha)^2$, we have either $\phi(+0) = 1$ (and hence $\phi(\alpha) = 1$ $\forall \alpha \in (0,1]$) or $\phi(+0) = 0$. If $\| \cdot \|$ is p-homogeneous, $\phi(\alpha) = \alpha^p$. For (s) and M and in general for a bounded norm we have $\phi(+0) = 1$, but the same is true also for the unbounded norm $\|x\| = \log(1+|x|)$ on the real axis. However, we have

THEOREM 4. *If for an F-space X*

$$\phi(+0) = 0$$

then X has the circle property with $r(O) = r(A) = r(M)$.

Proof. Since $\|\alpha x\| \leqq \phi(\alpha)\|x\|$, X has bounded spheres and is therefore locally bounded whence $r(O) = r(A)$. For $r(O) = r(M)$ it is sufficient to show that O at z_1 entails M in $|z| < |z_1|$. Indeed, O at z_1 implies $\|a_n z_1^n\| \to 0$ so that $\|a_n z_1^n\| \leqq c \; \forall n$. Now let $|z| < |z_1|$. We have

$$\sum_0^N \|a_n z^n\| = \sum_0^N \left\|a_n z_1^n \left(\frac{z}{z_1}\right)^n\right\| \leqq c \sum_0^N \phi\left(\left|\frac{z}{z_1}\right|^n\right).$$

The right hand side is bounded for $N \to \infty$ for every $|z| < |z_1|$ if and only if $\Sigma \phi(\alpha^n) < \infty \; \forall \alpha \in [0,1]$. Applying the integral criterion this is equivalent to

$$\int_0^1 \frac{\phi(\alpha)}{\alpha} d\alpha < \infty.$$

But there exists an $\alpha_0 \in (0,1)$ with $\phi(\alpha_0) = d < 1$, therefore $\phi(\alpha_0^n) \leqq d^n$ and

$$\int_0^1 \frac{\phi(\alpha)}{\alpha} d\alpha = \sum_{n=0}^{\infty} \int_{\alpha_0^{n+1}}^{\alpha_0^n} \frac{\phi(\alpha)}{\alpha} d\alpha \leqq \log\frac{1}{\alpha_0} \sum_{n=0}^{\infty} d^n < \infty,$$

q.e.d.

Theorems 3 and 4 apply, for instance, to L_p, l_p, and H_p $(0 < p < 1)$. In a locally convex F-space, we have in general only $r(O) = r(A) \geqq r(M)$ where the inequality may be strict as examples in (*s*) show.

We note that local convexity or local boundedness are only sufficient conditions for the circle property. For instance, $X = (L_p)^\infty$ $(0 < p < 1)$ is neither locally convex nor locally bounded, but has the circle property by Theorem 3 and Lemma 2.

6. ***F*-spaces without circle property.** We simulate in a general F-space what led to the construction of the counterexample in the non-atomic case of Theorem 1:

THEOREM 5. *Let X be an F-space. If there exist a constant $c > 0$ and for every $\varepsilon > 0$ a finite number $n = n(\varepsilon)$ of elements $x_1, \cdots, x_n \in X$ with*

$$\sup_{\alpha \in C} \|\alpha x_i\| \leqq \varepsilon \text{ for } i = 1, \cdots, n$$

and

$$\left\|\sum_1^n x_i\right\| \geqq c$$

then for every z_1, z_2 $(0 < |z_1| < |z_2|)$ *there exists a power series* $\Sigma a_n z^n$ *converging O at* $z = z_2$ *(to* $0 \in X$*) but diverging O at* $z = z_1$.

Proof. Let $\varepsilon_k \to 0$ and let $x_1^{(k)}, \cdots, x_{n_k}^{(k)}$ be the elements fulfilling the above conditions for $\varepsilon = \varepsilon_k$, set $h_0 = 0$, $h_1 = n_1$, $h_k = n_1 + \cdots + n_k$, $s_0 = 0 \in X$, and for $h_{k-1} < j \leqq h_k$, $k = 1, 2, \cdots$

$$s_j = \left(\frac{z_2}{z_1}\right)^j x_{j-h_{k-1}}^{(k)}.$$

Then the power series $\Sigma a_n z^n$ with $a_0 = 0$ and

$$a_n = z_2^{-n}(s_n - s_{n-1}) \qquad (n \geqq 1)$$

has the required property. Indeed

$$\sum_0^N a_n z^n = \left(\frac{z}{z_2}\right)^N s_N + \left(1 - \frac{z}{z_2}\right) \sum_0^{N-1} \frac{s_n}{z_2^n} z^n, \tag{3}$$

so that for $z = z_2$

$$\left\| \sum_0^N a_n z^n \right\| = \| s_N \| \leqq \varepsilon_k$$

for $h_{k-1} < N \leqq h_k$. Thus, $\Sigma a_n z^n$ converges O at $z = z_2$ because $\varepsilon_k \to 0$. On the other hand, if $\Sigma a_n z_1^n$ would converge O, so also, by (3) and $s_N(z_1/z_2)^N \to 0$ would $\Sigma s_n z_1^n z_2^{-n}$. This is impossible since by definition of the s_n's

$$\left\| \sum_{h_{k-1}+1}^{h_k} s_n z_1^n z_2^{-n} \right\| = \left\| \sum_{j=1}^{n_k} x_j^{(k)} \right\| \geqq c$$

and the space X is complete, q.e.d.

Theorem 5 characterizes a class of F-spaces containing *arbitrarily short straight lines* (i.e. for every neighborhood V of $0 \in X$ there corresponds some $x \neq 0$ for which $\alpha x \in V$ $\forall \alpha \in C$). These spaces are necessarily of infinite dimension. On the other hand, we have

THEOREM 6. *Every F-space having arbitrarily short straight lines contains an infinite-dimensional subspace which has the circle property with* $r(O) = r(A)$.

Proof. By Theorem 9 of [3], an arbitrary F-spaces has arbitrarily short straight lines if and only if it contains a subspace isomorphic to (s). But (s) has the circle property, with $r(O) = r(A)$, q.e.d.

REFERENCES

1. L. Arnold, *Über die Konvergenz einer zufälligen Potenzreihe*, J. Reine Angew. Math. **222** (1966), 79–112.

2. L. Arnold, *Random Power Series*, Statistical Laboratory Publications No. 1 (1966), Michigan State University, East Lansing, Michigan.

3. C. Bessaga, A. Pełczyński and S. Rolewicz, *Some Properties of the Space* (*s*), Colloq. Math. 7 (1959), 45–51.

4. N. Dunford and J. T. Schwartz, *Linear Operators*, Part 1, New York, 1958.

5. A. Dvoretzky, *On series in linear topological spaces*, Israel J. Math. 1 (1963), 37–347.

6. J. L. Kelley and I. Namioka, *Linear Topological Spaces*, Princeton, N. J., 1963.

7. R. S. Phillips, *Integration in a convex linear topological space*, Trans. Amer. Math. Soc. **47** (1940), 114–145.

8. S. Rolewicz, *On a certain class of linear metric spaces*, Bull. Acad. Pol. Sci. Cl. III, **5** (1957), 471–473.

9. N. Wiener, *Note on a paper of M. Banach*, Fund. Math. **4** (1923), 136–143.

Michigan State University, East Lansing
and
Technische Hochschule, Stuttgart